CURRENT TRENDS IN QBF SOLVING

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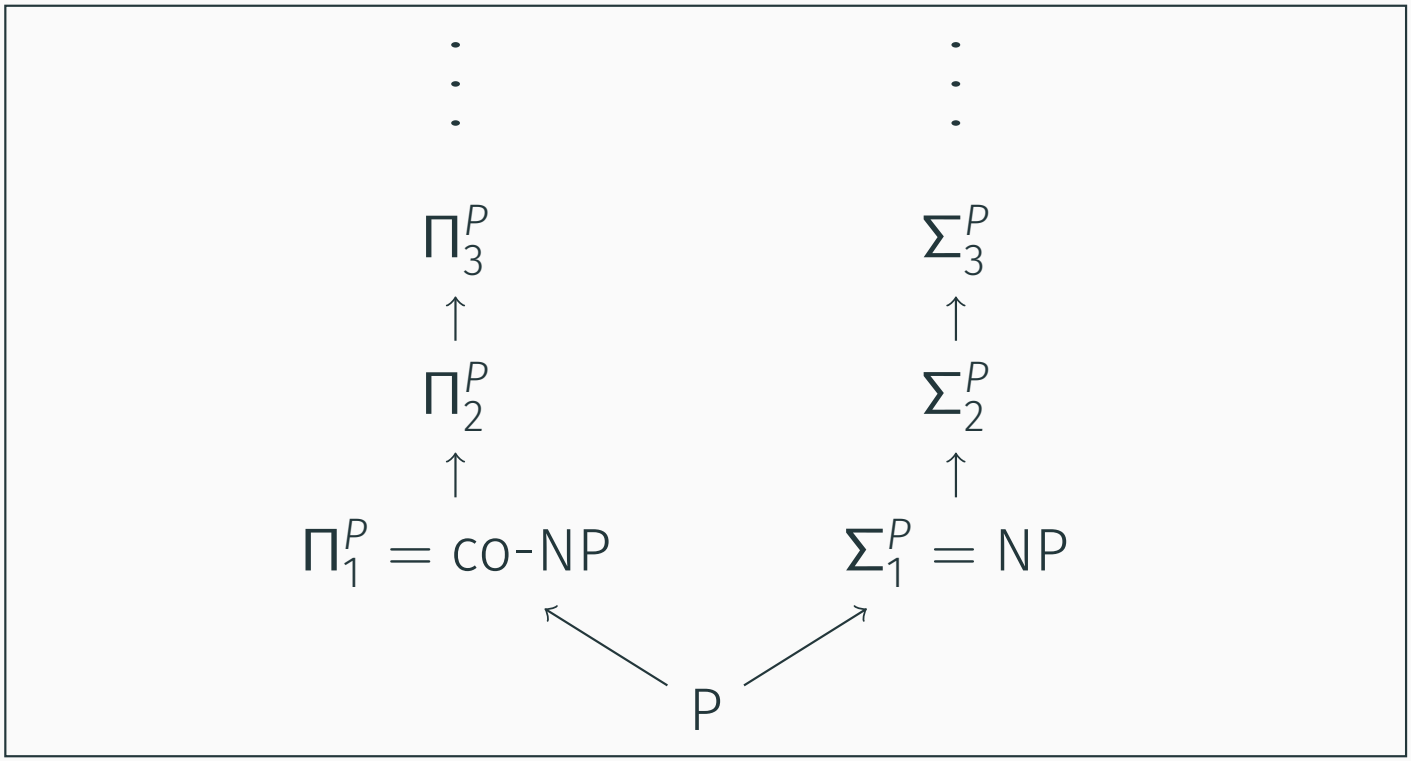
  (4) 1 (True)

RELATION TO COMPLEXITY THEORY



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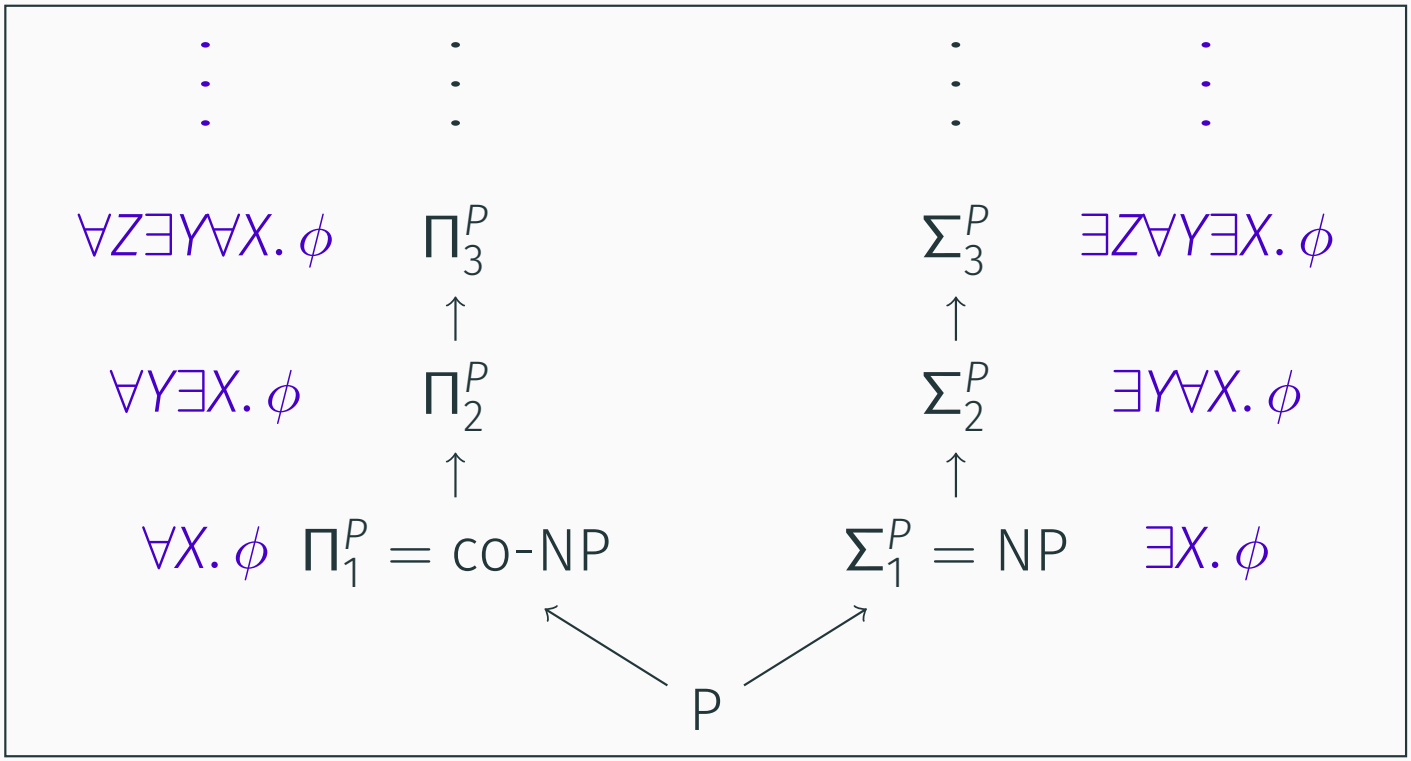

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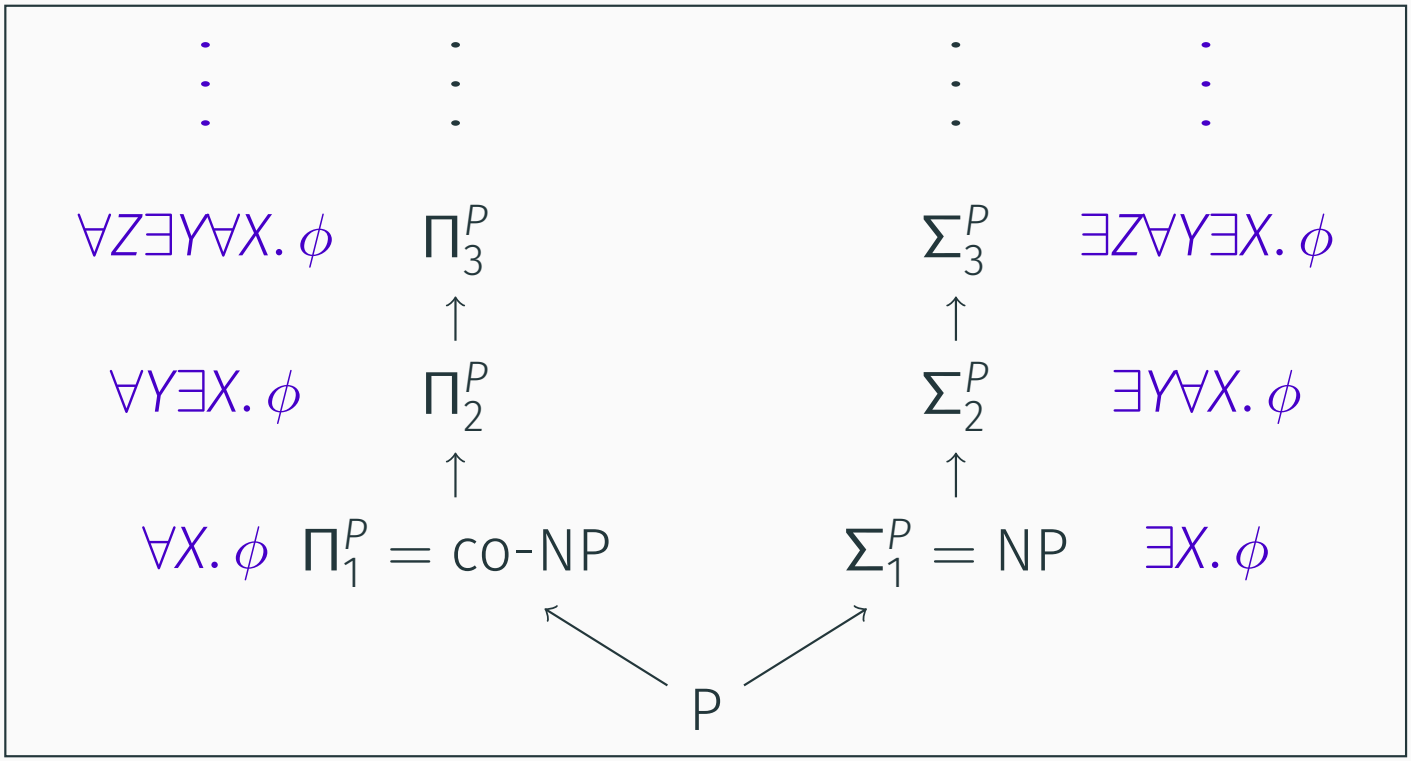

- Deciding QBF is PSPACE complete

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- A QBF is true iff there exists a winning strategy for $\exists$.

  Example

$$\forall u \exists e . (u \leftrightarrow e)$$

$\exists$-player wins by playing $e \triangleq u$.

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- In other reasoners?
  - SMT (e.g. Quantified bit vectors)
  - optimization with quantification (“MaxQBF”)
  - . . .

EXAMPLE: SMALLEST MUS

Given a CNF ϕ, construct the following QBF.

$$\exists S \forall X. \neg \left(\bigwedge_{C \in \phi} (\neg s_C \vee C) \right) \wedge |S| \leq k$$

Where

- $S = \{s_C \mid C \in \phi\}$ are fresh variables
- X are the original variables of ϕ
- $k \in \mathbb{N}$

[Ignatiev et al., 2015]

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- If $x_1 = x_{k+1} = 0$, then $\exists$ must play $x_2 = 1$.
- As otherwise $\forall$ would win by setting $x_k = 0$.

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Unification for the 2 players: [Zhang, 2006] [Klieber, 2014] [Goultiaeva et al., 2013]

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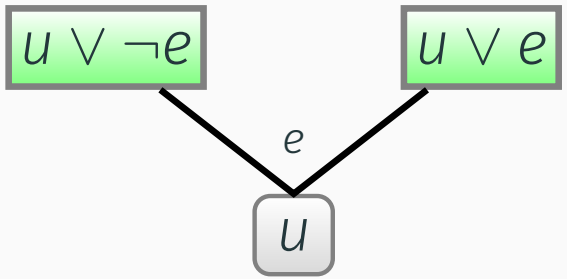

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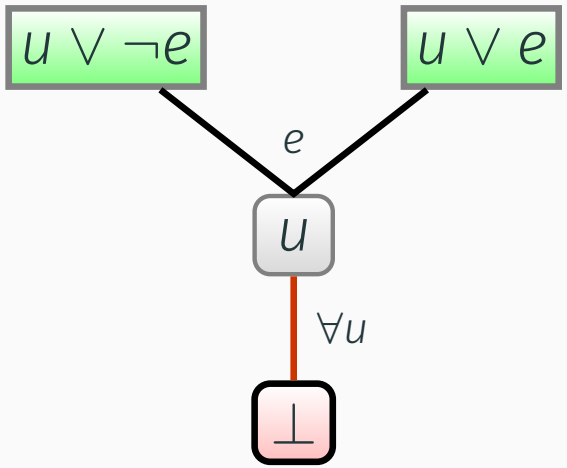

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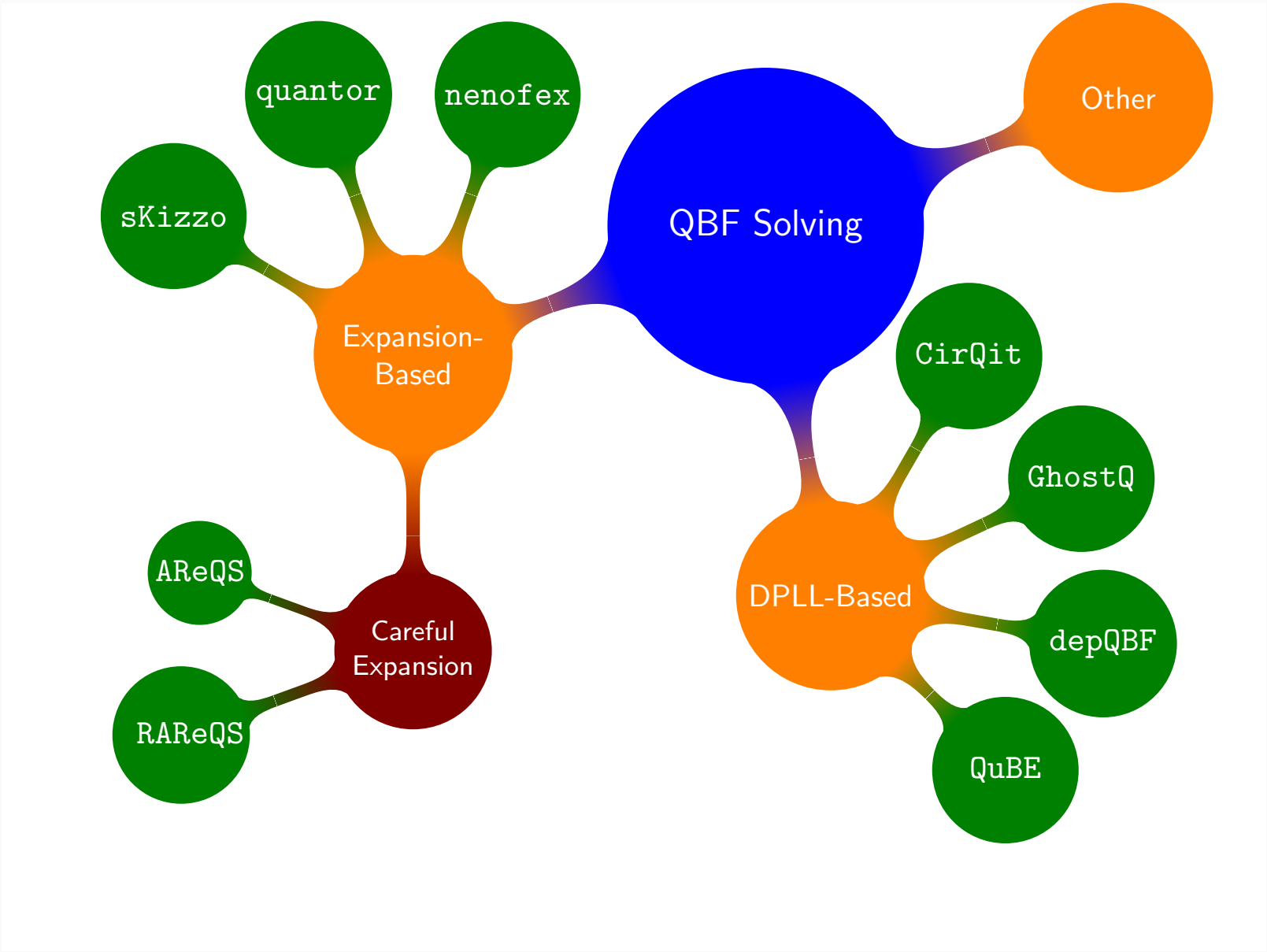
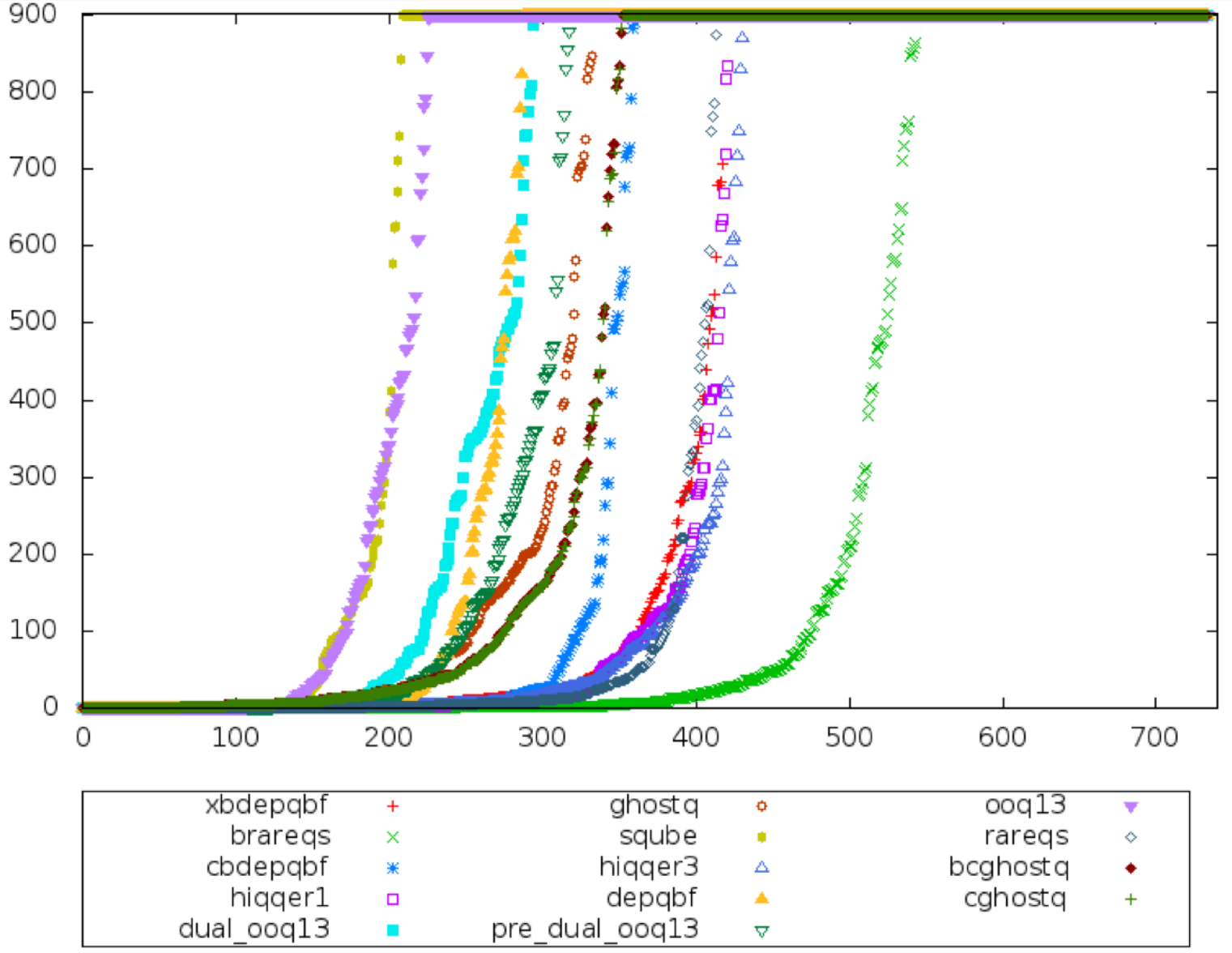
- $\mathsf{SAT}(\phi[\mu_0] \wedge \phi[\mu_1]) = \tau_2$ assignment to $\mathcal{E}$

OVERVIEW OF QBF SOLVERS



RESULTS, QBF-GALLERY '14, APPLICATION TRACK



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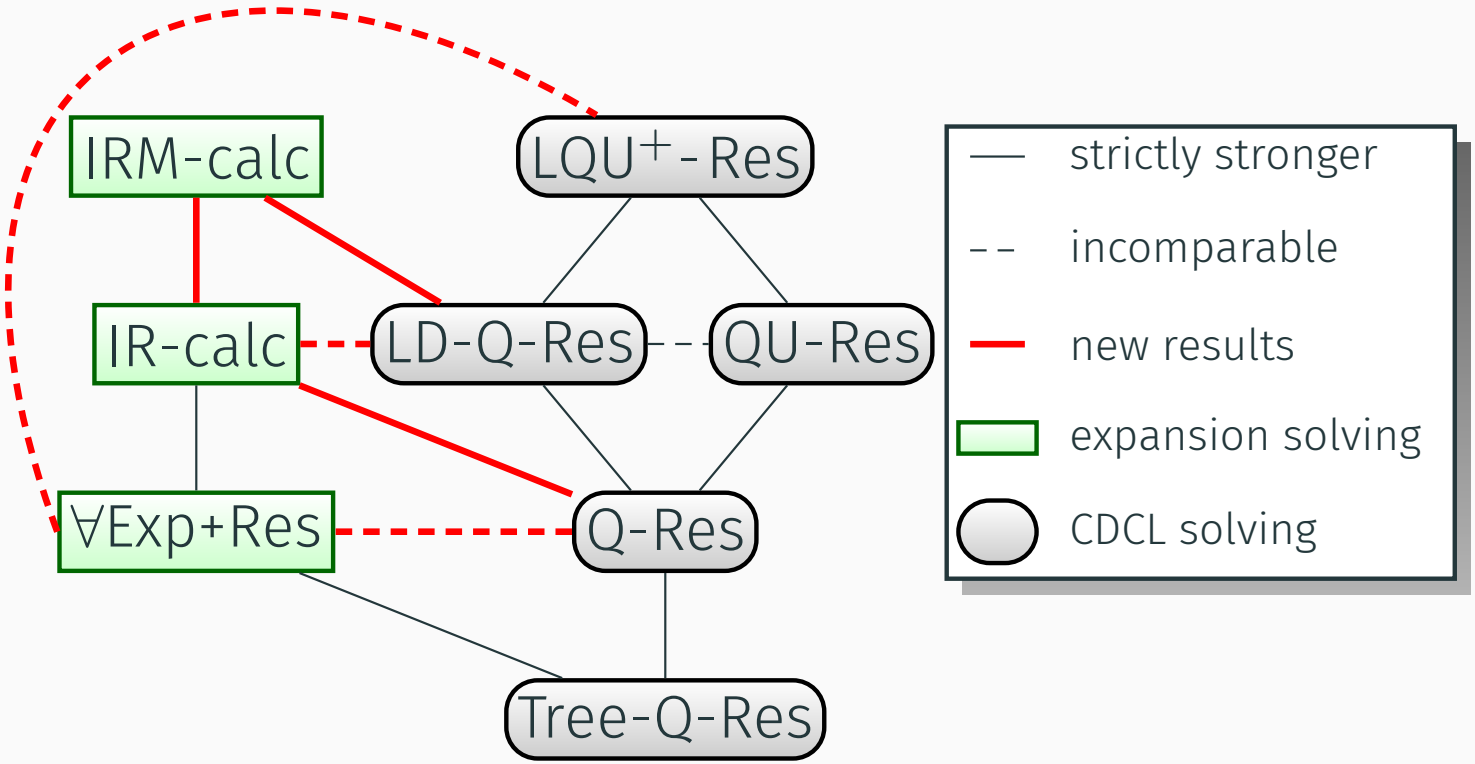
- These calculi are incomparable [Beyersdorff et al., 2015].

CALCULI ZOO [BEYERSDORFF ET AL., 2015]



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- Challenge: There are calculi with no corresponding solvers.
- Challenge: There are formula with easy strategies but that are hard to solve. How to look for strategies? [Bjørner et al., 2015]
- Challenge: How to make QBF more attractive, more theories? [Bjørner and Janota, 2015]

Thank You for Your Attention!

Questions?


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